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Vibration Analysis of Thin-Walled Curved Beams Using DQM

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Abstract

The differential quadrature method (DQM) is applied to computation of the eigenvalues of small amplitude free vibration for thin-walled curved beams including a warping contribution. Natural frequencies are calculated for single-span, curved, wide-flange uniform beams having a range of nondimensional parameters representing variations in warping stiffness, torsional stiffness, radius of curvature, included angle of the curve, polar mass moment of inertia, and various end conditions. Results are compared with existing exact and numerical solutions by other methods for cases in which they are available. It is found that the DQM gives good accuracy even when only a limited number of grid points is used. In addition, results are given for a cantilever beam not previously considered for this problem. Finally, parametric results are presented in dimensionless form.

Keywords: Differential Quadrature Method (DQM); Frequency; Numerical solution; Thin-walled curved beam; Vibration; Warping

1. Introduction

Horizontally curved beams are used frequently in highway bridge structures. Curved alignments of highway bridges and interchanges have been necessary for the smooth dissemination of traffic in large urban areas. The construction cost and time of curved beams associated with the substructure have been found to be significantly reduced by the use of curved beams. Furthermore, the construction time is a factor of immense importance in the selection of a suitable structural system where the construction site needs to be used for other operations during the construction period described by Kang and Yoo (1994). Owing to their importance in many fields of technology and engineering, the vibration behavior of a thin-walled curved beam has been the subject of a large number of investigations. Despite of a number of advantages, a curved member behaves in an extremely complex manner as compared to a straight member, and practicing engineers have often been discouraged by the complexity because of the initial curvature. However, the mathematical difficulties associated with curved members have been largely overcome with the application of digital computers and the development of numerical methods. Solutions of relevant differential equations have traditionally been obtained by the standard Rayleigh-Ritz, finite difference, or finite element methods (FEM).

The early investigators into the in-plane vibration of rings were Hoppe (1871) and Love (1944). Love (1944) improved on Hoppe's theory by allowing for stretching of the ring. Lamb (1888) investigated the statics of a curved bar with various boundary conditions and the dynamics of an incomplete free-free ring of small curvature. Den Hartog (1928) used the Rayleigh-Ritz method for finding the lowest natural frequency of circular arcs with clamped ends, and his work was extended by Volterra and Morell

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(1961) for the vibration of arches having center lines in the form of cycloids, catenaries, or parabolas.

Out-of-plane vibrations of complete and incomplete rings have been the subject of interest for several research workers. Ojalvo (1962) obtained the equations governing three-dimensional linear motions of elastic rings and results for generalized loadings and viscous damping using classical beam theory assumptions for the clamped ends. Culver (1967) and Shore and Chaudhuri (1972) studied the free vibration of horizontally curved beams using closed-form solutions of the equations of motion. Tan and Shore (1968) calculated the dynamic response of a single-span curved beam to moving loads. Chaudhuri and Shore (1977) studied the free vibration of horizontally curved beams using the finite element method. Snyder and Wilson (1992) calculated the free vibration frequencies of continuous horizontally curved beams using a non-explicit closed-form solution ofthe partial differential equations of motion.

The differential quadrature method (DQM) is a numerical technique of rather recent origin, which by its continually growing applications in a variety of problems of engineering and physical sciences, is a competing alternative to the conventional numerical techniques for the solution of initial and boundary value problems. By formulating the quadrature rule for a derivative as an analogous extension of quadrature for integrals, Bellman and Casti (1971) proposed the DQM as a new technique for the numerical solution of initial value problems of ordinary and partial differential equations. It was applied for the first time to the static analysis of structural components by Jang et al. (1989). The versatility of the DQM to engineering analysis in general and to structural analysis in particular is becoming increasingly evident by the number of related publications in recent years. Some recent works (Malik and Bert, 1994; Malik and Civan, 1995) have focused on the assessment of the numerical accuracy and computational efficiency of the DQM. It has been shown that in both these respects the DQM stands out in comparison to the conventional numerical solution techniques of the finite difference and finite element methods. Recently, Kang and Han (1998) applied the method to classical and shear deformable theories of circular curved beams, and Kang and Kim (2002) studied the extensional vibration analysis of curved beams using the DQM.

In the present work, the DQM is used to analyze the free vibration behavior of a single-span, curved, wide-flange beam including a warping contribution but neglecting the transverse shearing deformation. The frequencies are calculated for the member over a range of nondimensional parameters representing variations in the warping stiffness, torsional stiffness, radius of curvature, included angle of the curve, and polar mass moment of inertia. The differential equations used to model the elastic behavior of the beam, derived by Vlasov (1961), are based on the assumption that the cross-sectional shape is constant along the entire length of the member and doubly symmetric; i.e., the shear center and centroid coincide. Various boundary conditions are considered. Numeri-cal results are compared with existing exact solutions and numerical solutions by the Rayleigh-Ritz and the FEM.

2. Governing differential equations

The uniform curved beam considered is shown in Fig. 1. A point on the centroidal axis is defined by the angle θ, measured from the left support. The tangential and radial displacements of the beam axis are w and u, respectively. Here, v is the displacement at right angle to the plane of the beam, R is the radius of the centroidal axis, and ϕ is the angular rotation of a cross section of the principal axes about the tangential axis. These displacements are considered to be positive in the directions indicated. A mathematical study of the out-of-plane vibration of the curved beam of small cross section is carried out starting with the basic equations of motiongiven by Ojalvo (1962). If the effects of rotatory inertia, warping, and shear deformation are neglected, the differential equation governing the coupled twist-bending vibration of the thin curved beam can be written as

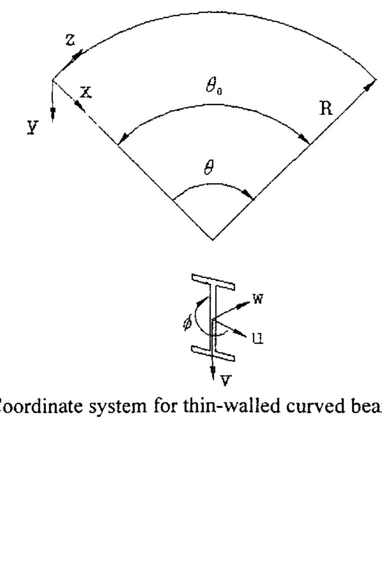

Fig. 1. Coordinate system for thin-walled curved beam.

$$EI_x \frac{\partial^4 \upsilon}{\partial z^4} - \frac{GK_T}{R^2}\frac{\partial^2 \upsilon}{\partial z^2} - \frac{EI_x + GK_T}{R}\frac{\partial^2 \phi}{\partial z^2} + m\frac{\partial^2 \upsilon}{\partial t^2} = 0 \quad (1)$$

$$-\frac{EI_x + GK_T}{R}\frac{\partial^2 \upsilon}{\partial z^2} - GK_T\frac{\partial^2 \phi}{\partial z^2} + \frac{EI_x}{R^2}\phi = 0 \quad (2)$$

where G is the shear modulus, K_T is the Saint-Venant torsion constant, I_x is the area moment of inertia of the cross section, E is the Young's modulus of elasticity, and m is the mass per unit length.

The differential equations governing the thin-walled curved beam including the rotatory inertia and the warping contribution but neglecting the shear deformation can be written as (Snyder and Wilson, 1992)

$$\left(\frac{EI_w}{R^2} + EI_x\right)\frac{\partial^4 \upsilon}{\partial z^4} - \frac{GK_T}{R^2}\frac{\partial^2 \upsilon}{\partial z^2} + \frac{EI_w}{R}\frac{\partial^4 \phi}{\partial z^4} - \frac{EI_x + GK_T}{R}\frac{\partial^2 \phi}{\partial z^2} + m\frac{\partial^2 \upsilon}{\partial t^2} = 0 \quad (3)$$

$$\frac{EI_w}{R}\frac{\partial^4 \upsilon}{\partial z^4} - \frac{EI_x + GK_T}{R}\frac{\partial^2 \upsilon}{\partial z^2} + EI_w\frac{\partial^4 \phi}{\partial z^4} - GK_T\frac{\partial^2 \phi}{\partial z^2} + \frac{EI_x}{R^2}\phi + mr^2\frac{\partial^2 \phi}{\partial t^2} = 0 \quad (4)$$

where I_w is the warping constant, and r is the polar radius of gyration.

To find the corresponding free vibration frequencies, the following normal-mode solutions are assumed:

$$\upsilon(z,t) = V(z)\sin \omega t, \quad \phi(z,t) = \Phi(z)\sin \omega t \quad (5)$$

Replacing z by $R\theta$ and using Eq. (5), one can rewrite Eqs. (3) and (4) as

$$\left(\frac{EI_w}{R^2} + EI_x\right)\frac{V^{IV}}{R^4\theta_0^4} - \frac{GK_T}{R^2}\frac{V''}{R^2\theta_0^2} + \frac{EI_w}{R}\frac{\Phi^{IV}}{R^4\theta_0^4} - \frac{EI_x + GK_T}{R}\frac{\Phi''}{R^2\theta_0^2} - m\omega^2 V = 0 \quad (6)$$

$$\frac{EI_w}{R}\frac{V^{IV}}{R^4\theta_0^4} - \frac{EI_x + GK_T}{R}\frac{V''}{R^2\theta_0^2} + EI_w\frac{\Phi^{IV}}{R^4\theta_0^4} - GK_T\frac{\Phi''}{R^2\theta_0^2} + \frac{EI_x}{R^2}\Phi - mr^2\omega^2\Phi = 0 \quad (7)$$

where each prime denotes one differentiation with respect to the dimensionless distance coordinate $X=\theta/\theta_0$, in which θ_0 is the opening angle of the member.

Now the following dimensionless parameters are introduced:

$$\xi = r/R, \ \bar{C} = GK_T/(EI_x), \ \bar{D} = I_w/(I_x R^2), \quad \bar{\omega} = (m(R\theta_0)^4/(EI_x))^{1/2}\omega \quad (8)$$

Thus, one can rewrite Eqs. (6) and (7) in non-dimensional form as

$$(1+\bar{D})V^{IV} - \theta_0^2\bar{C}V'' + R\bar{D}\Phi^{IV} - R\theta_0^2(1+\bar{C})\Phi'' - \bar{\omega}^2 V = 0 \quad (9)$$

$$\frac{1}{R\xi^2}\bar{D}V^{IV} - \frac{\theta_0^2}{R\xi^2}(1+\bar{C})V'' + \frac{1}{\xi^2}\bar{D}\Phi^{IV} - \frac{\theta_0^2}{\xi^2}\bar{C}\Phi'' + \frac{\theta_0^4}{\xi^2}\Phi - \bar{\omega}^2\Phi = 0 \quad (10)$$

The following boundary conditions are taken for simply supported ends (Tan and Shore, 1968): (a) no vertical deflection; (b) no torsional rotation; (c) no bending moment; and (d) no bimoment. The bending moment and the bimoment of the beam can be written as

$$M_x = EI_x\left(\frac{\phi}{R} - \frac{d^2\upsilon}{dz^2}\right), \ B_w = -EI_w\left(\frac{d^2\phi}{dz^2} + \frac{1}{R}\frac{d^2\upsilon}{dz^2}\right) \quad (11)$$

For clamped end v, ϕ, dv/dz, and τ equal zero where τ represents the warping as defined by Vlasov (1961). It can be written as

$$\tau(z) = -\left(\frac{1}{R}\frac{dv}{dz} + \frac{d\phi}{dz}\right) \quad (12)$$

The following boundary conditions are taken for free ends: (a) no bending moment; (b) no bimoment; (c) no torsional moment; and (d) no transverse shear force.

The torsional moment T and the transverse shear force Q of the beam can be written as

$$T = GK_T\left(\frac{d\phi}{dz} + \frac{dv}{Rdz}\right) - EI_w\left(\frac{d^3\phi}{dz^3} + \frac{d^3v}{Rdz^3}\right), \quad Q = \frac{T}{R} + \frac{\partial M_x}{\partial z} \quad (13)$$

The boundary conditions for simply supported, clamped, and free ends are, respectively

$$V = \Phi = V'' = \Phi'' = 0 \quad (14)$$

$$V = \Phi = V' = \Phi' = 0 \quad (15)$$

$$\Phi - \frac{V''}{R\theta_0^2} = -\Phi'' - \frac{V''}{R} = \bar{C}(\Phi' + \frac{V'}{R}) - \bar{D}(\frac{\Phi'''}{\theta_0^2} + \frac{V'''}{R\theta_0^2})$$
$$= \frac{\Phi'}{R} - \frac{V'''}{R^2\theta_0^2} + \bar{C}(\Phi' + \frac{V'}{R}) - \bar{D}(\frac{\Phi'''}{\theta_0^2} + \frac{V'''}{R\theta_0^2})$$
$$= 0 \quad (16)$$

3. Differential quadrature method

From a mathematical point of view, the application of the differential quadrature method to a partial differential equation can be expressed as follows:

$$L\{f(x)\}_i = \sum_{j=1}^{N} W_{ij}\, f(x_j) \quad for \quad i, j = 1,2,\ldots\ldots,N \quad (17)$$

where L denotes a differential operator, x_j are the discrete points considered in the domain, $f(x_j)$ are the function values at these points, W_{ij} are the weighting coefficients attached to these function values, and N denotes the number of discrete points in the domain. This equation, thus, can be expressed as the derivatives of a function at a discrete point in terms of the function values at all discrete points in the variable domain (Kang and Han, 1998).

The general form of the function $f(x)$ is taken as

$$f_k(x) = x^{k-1} \quad for \quad k = 1,2,\ldots.,N \quad (18)$$

If the differential operator L represents an n^{th} derivative, then

$$\sum_{j=1}^{N} W_{ij} x_j^{k-1} = (k-1)(k-2)\cdots(k-n)x_i^{k-n-1}$$
$$for \quad i,k = 1,2,\ldots.,N \quad (19)$$

This expression represents N sets of N linear algebraic equations, giving a unique solution for the weighting coefficients, W_{ij}, since the coefficient matrix is a Vandermonde matrix, which always has an inverse as described by Hamming (1973).

4. Application

Applying the differential quadrature method to Eqs. (9) and (10) gives

$$(1+\bar{D})\sum_{j=1}^{N} D_{ij}V_j - \theta_0^2\bar{C}\sum_{j=1}^{N} B_{ij}V_j$$
$$+R\bar{D}\sum_{j=1}^{N} D_{ij}\Phi_j - R\theta_0^2(1+\bar{C})\sum_{j=1}^{N} B_{ij}\Phi_j \quad (20)$$
$$-\bar{\omega}^2 V_i = 0$$

$$\frac{1}{R\xi^2}\bar{D}\sum_{j=1}^{N} D_{ij}V_j - \frac{\theta_0^2}{R\xi^2}(1+\bar{C})\sum_{j=1}^{N} B_{ij}V_j$$
$$+\frac{1}{\xi^2}\bar{D}\sum_{j=1}^{N} D_{ij}\Phi_j - \frac{\theta_0^2}{\xi^2}\bar{C}\sum_{j=1}^{N} B_{ij}\Phi_j \quad (21)$$
$$+\frac{\theta_0^4}{\xi^2}\Phi_i - \bar{\omega}^2\Phi_i = 0$$

where B_{ij} and D_{ij} are the weighting coefficients for the second- and the fourth-order derivatives, respectively, along the dimensionless axis.

The boundary conditions for simply supported ends, given by Eq. (14), can be expressed in differential quadrature form as follows:

$$V_1 = 0 \quad \text{and} \quad \Phi_1 = 0 \qquad \text{at } X = 0$$
$$\sum_{j=1}^{N} B_{2j}V_j = 0 \quad \text{and} \quad \sum_{j=1}^{N} B_{2j}\Phi_j = 0 \qquad \text{at } X = 0+\delta$$
$$\sum_{j=1}^{N} B_{(N-1)j}V_j = 0 \quad \text{and} \quad \sum_{j=1}^{N} B_{(N-1)j}\Phi_j = 0 \qquad \text{at } X = 1-\delta$$
$$V_N = 0 \quad \text{and} \quad \Phi_N = 0 \qquad \text{at } X = 1 \quad (22)$$

where δ denotes a very small dimensionless distance measured from the boundary ends of the member.

The boundary conditions for clamped ends, given by Eq. (15), can be expressed in differential quadrature form as follows:

$$V_1 = 0 \quad \text{and} \quad \Phi_1 = 0 \qquad \text{at } X = 0$$
$$\sum_{j=1}^{N} A_{2j}V_j = 0 \quad \text{and} \quad \sum_{j=1}^{N} A_{2j}\Phi_j = 0 \qquad \text{at } X = 0+\delta$$
$$\sum_{j=1}^{N} A_{(N-1)j}V_j = 0 \quad \text{and} \quad \sum_{j=1}^{N} A_{(N-1)j}\Phi_j = 0 \qquad \text{at } X = 1-\delta$$
$$V_N = 0 \quad \text{and} \quad \Phi_N = 0 \qquad \text{at } X = 1 \quad (23)$$

The mixed boundary conditions for one simply supported and one clamped end, given by Eqs. (14) and (15), can be expressed in differential quadrature form as follows:

$$V_1 = 0 \quad \text{and} \quad \Phi_1 = 0 \qquad \text{at } X = 0$$
$$\sum_{j=1}^{N} B_{2j}V_j = 0 \quad \text{and} \quad \sum_{j=1}^{N} B_{2j}\Phi_j = 0 \qquad \text{at } X = 0+\delta$$

$$\sum_{j=1}^{N} A_{(N-1)j} V_j = 0 \text{ and } \sum_{j=1}^{N} A_{(N-1)j} \Phi_j = 0 \quad \text{at } X = 1-\delta$$

$$V_N = 0 \text{ and } \Phi_N = 0 \quad \text{at } X = 1 \tag{24}$$

The boundary conditions for one clamped and one free end, given by Eqs. (15) and (16), can be expressed in differential quadrature form as follows:

$$V_1 = 0 \text{ and } \Phi_1 = 0 \quad \text{at } X = 0$$

$$\sum_{j=1}^{N} A_{2j} V_j = 0 \text{ and } \sum_{j=1}^{N} A_{2j} \Phi_j = 0 \quad \text{at } X = 0+\delta$$

$$\bar{C}\left(\sum_{j=1}^{N} A_{(N-1)j} \Phi_j + \frac{\sum_{j=1}^{N} A_{(N-1)j} V_j}{R}\right) - \bar{D}\left(\frac{\sum_{j=1}^{N} C_{(N-1)j} \Phi_j}{\theta_0^2} + \frac{\sum_{j=1}^{N} C_{(N-1)j} V_j}{R\theta_0^2}\right) = 0$$

and $$\frac{\sum_{j=1}^{N} A_{(N-1)j} \Phi_j}{R} - \frac{\sum_{j=1}^{N} C_{(N-1)j} V_j}{R^2\theta_0^2} + \bar{C}\left(\sum_{j=1}^{N} A(N-1)j\Phi j + \frac{\sum_{j=1}^{N} A_{(N-1)j} V_j}{R}\right) - \bar{D}\left(\frac{\sum_{j=1}^{N} C_{(N-1)j} \Phi_j}{\theta_0^2} + \frac{\sum_{j=1}^{N} C_{(N-1)j} V_j}{R\theta_0^2}\right) = 0 \quad \text{at } X = 1-\delta$$

$$\Phi N - \frac{\sum_{j=1}^{N} B_{Nj} V_j}{R\theta_0^2} = \text{and } -\sum_{j=1}^{N} B_{Nj} \Phi_j - \frac{\sum_{j=1}^{N} B_{Nj} V_j}{R} = 0$$

$$\text{at } X = 1 \tag{25}$$

where $C_{(N-1)j}$ are the weighting coefficients for the third-order derivatives along the dimensionless axis.

Those governing equations with the appropriate boundary conditions can be solved to obtain the natural frequencies for a single-span, curved, wide-flange uniform beam.

5. Numerical results and comparisons

The natural frequencies of the out-of-plane vibration of a curved beam are calculated by the differential quadrature method (DQM), and the results are presented together with existing exact solutions and numerical solutions by the Rayleigh-Ritz and the finite element method (FEM).

The frequency parameter $\bar{\omega}$ is evaluated for the case of a single-span, horizontally curved, thin-walled beam over a range of nondimensional parameters representing variations in warping stiffness, torsional stiffness, radius of curvature, included angle of the curve, and polar mass moment of inertia with various boundary conditions.

The first example considered here has $\bar{C} = 0.05$ and 2.00, and ξ =0.005 and 0.020. Theses values were selected to match upper and lower limits for elevated guideways reported by Wilson et al. (1985). The parameter $\bar{D}$ is allowed to vary from 0 to 0.1 (Snyder and Wilson, 1992).

Table 1 presents the results of convergence studies relative to the number of grid point N and the δ parameter with θ_0 =15° and 60°. The data show that the accuracy of the numerical solution increases with increasing N. Then, numerical instabilities arise if N becomes too large (possibly not greater than 19). The optimal value for N is found to be 11~15, which is obtained from trial-and-error calculations. Table 1 also shows the sensitivity of the numerical solution to the choice of δ. The solution accuracy decreases due to numerical insta-bilities if δ becomes too big (possible not greater than 0.001). The optimal value for δ is found to be 1×10^{-6}~1×10^{-8}. The convergence and the accuracy of the quadrature solutions depend largely on the accuracy of the weighting coefficients obtained by the number of gird points and the spacingof grid points. From the above analysis, the numerical results of convergence and accuracy are fairly good when the results are computed with 11 discrete points along the dimen-sionless X-axis with δ $=1\times10^{-6}$. It is observed that the maximum difference between the exact and the DQM solutions is less than 0.1 %. Subsequently, 11 discrete points along the dimensionless X-axis with $\delta=1\times10^{-6}$ will be used in the following analysis.

In Tables 2 and 3, the exact solutions by Culver (1967) are compared with those by the DQM for the case of both ends simply supported. It is seen that excellent agreement is achieved between the DQM solutions and the analytical solutions. The available numerical results are also utilized for evaluating the accuracy of the DQM in the solution of this problem. Culver (1967) determined the fundamental natural frequency parameters of the member using the Rayleigh-Ritz method for the cases of clamped ends and mixed simply supported-clamped ends. The results are summarized in Tables 4~7. From Tables

Table 1. Fundamental frequency parameter $\bar{\omega}$ for free vibration of a thin-walled curved beam with simply supported ends including a range of grid point N and δ, $\bar{C}$=0.05, $\bar{D}$=0.1, and ξ = 0.005.

θ_0	Exact (Culver, 1967)	N	DQM			
			δ			
			1x10^{-2}	1x10^{-4}	1x10^{-6}	1x10^{-8}
15°	0.9928	7	1.0179	0.9978	0.9976	0.9976
		9	1.0135	0.9939	0.9927	0.9927
		11	1.0136	0.9928	0.9928	0.9928
		13	1.0136	0.9928	0.9928	0.9928
		15	1.0136	0.9928	0.9928	0.9928
60°	0.8410	7	0.8725	0.8474	0.8472	0.8472
		9	0.8669	0.8411	0.8409	0.8409
		11	0.8670	0.8413	0.8410	0.8410
		13	0.8670	0.8413	0.8410	0.8410
		15	0.8670	0.8413	0.8410	0.8410

Table 2. Fundamental frequency parameter $\bar{\omega}$ for free vibration of a thin-walled curved beam with simply supported ends for N = 11 and δ=1x10^{-6}; θ_0 = 15°.

θ_0	$\bar{C}$	ξ	$\bar{D}$	Exact (Culver, 1967)	DQM
15°	0.05	0.005	0	0.9264	0.9264
			0.001	0.9753	0.9753
			0.1	0.9928	0.9928
		0.02	0	0.7936	0.7936
			0.001	0.9665	0.9665
			0.1	0.9926	0.9926
	2.0	0.005	0	0.9913	0.9913
			0.001	0.9914	0.9914
			0.1	0.9929	0.9928
		0.02	0	0.9909	0.9909
			0.001	0.9910	0.9910
			0.1	0.9926	0.9926

Table 3. Fundamental frequency parameter $\bar{\omega}$ for free vibration of a thin-walled curved beam with simply supported ends for N = 11 and δ=1x10^{-6}; θ_0= 45°.

θ_0	$\bar{C}$	ξ	$\bar{D}$	Exact (Culver, 1967)	DQM
45°	0.05	0.005	0	0.6243	0.6243
			0.001	0.6713	0.6713
			0.1	0.9202	0.9202
		0.02	0	0.6142	0.6142
			0.001	0.6626	0.6626
			0.1	0.9198	0.9198
	2.0	0.005	0	0.9232	0.9232
			0.001	0.9233	0.9233
			0.1	0.9295	0.9295
		0.02	0	0.9228	0.9228
			0.001	0.9229	0.9229
			0.1	0.9292	0.9292

Table 4. Fundamental frequency parameter $\bar{\omega}$ for free vibration of a thin-walled curved beam with clamped ends for N = 11 and δ=1x10^{-6}; θ_0=15°.

θ_0	$\bar{C}$	ξ	$\bar{D}$	Rayleigh-Ritz solution (Culver, 1967)	DQM
15°	0.05	0.005	0	2.2156	2.2153
			0.001	2.2600	2.2599
			0.1	2.2638	2.2630
		0.02	0	1.0726	1.0238
			0.001	2.2557	2.2556
			0.1	2.2626	2.2626
	2.0	0.005	0	2.2621	2.2621
			0.001	2.2623	2.2625
			0.1	2.2632	2.2630
		0.02	0	2.2610	2.2610
			0.001	2.2615	2.2514
			0.1	2.2626	2.2626

Table 5. Fundamental frequency parameter $\bar{\omega}$ for free vibration of a thin-walled curved beam with clamped ends for N = 11 and δ=1x10^{-6}; θ_0 = 45°.

θ_0	$\bar{C}$	ξ	$\bar{D}$	Rayleigh-Ritz solution (Culver, 1967)	DQM
45°	0.05	0.005	0	2.0673	2.0550
			0.001	2.1351	2.1316
			0.1	2.2308	2.2307
		0.02	0	2.0150	2.0064
			0.001	2.1144	2.1111
			0.1	2.2303	2.2301
	2.0	0.005	0	2.2254	2.2254
			0.001	2.2257	2.2256
			0.1	2.2314	2.2312
		0.02	0	2.2245	2.2245
			0.001	2.2248	2.2247
			0.1	2.2308	2.2307

Table 6. Fundamental frequency parameter $\bar{\omega}$ for free vibration of a thin-walled curved beam with one simply supported and one clamped end for N = 11 and δ=1x10^{-6}; θ_0=15°.

θ_0	$\bar{C}$	ξ	$\bar{D}$	Rayleigh-Ritz solution (Culver, 1967)	DQM
15°	0.05	0.005	0	1.5027	1.5028
			0.001	1.5494	1.5494
			0.1	1.5568	1.5569
		0.02	0	1.0033	0.9685
			0.001	1.5431	1.5431
			0.1	1.5565	1.5566
	2.0	0.005	0	1.5555	1.5557
			0.001	1.5557	1.5559
			0.1	1.5567	1.5569
		0.02	0	1.5549	1.5550
			0.001	1.5551	1.5552
			0.1	1.5565	1.5566

Table 7. Fundamental frequency parameter $\bar{\omega}$ for free vibration of a thin-walled curved beam with one simply supported and one clamped end for $N = 11$ and $\delta=1\times10^{-6}$; $\theta_0=45^\circ$.

θ_0	$\bar{C}$	ξ	$\bar{D}$	Rayleigh-Ritz solution (Culver, 1967)	DQM
45°	0.05	0.005	0	1.2899	1.2731
			0.001	1.3488	1.3426
			0.1	1.5097	1.5096
		0.02	0	1.2635	1.2480
			0.001	1.3220	1.3261
			0.1	1.5092	1.5092
	2.0	0.005	0	1.5054	1.5054
			0.001	1.5056	1.5056
			0.1	1.5120	1.5120
		0.02	0	1.5048	1.5048
			0.001	1.5050	1.5050
			0.1	1.5116	1.5116

Table 8. Fundamental frequency parameter $\bar{\omega}$ for free vibration of a thin-walled curved beam with one clamped and one free end for $N = 11$ and $\delta=1\times10^{-6}$; $\theta_0=15^\circ$.

θ_0	$\bar{C}$	ξ	$\bar{D}$	DQM
15°	0.05	0.005	0	0.9676
			0.001	0.9943
			0.1	0.9972
		0.02	0	0.9169
			0.001	0.9934
			0.1	0.9971
	2.0	0.005	0	0.9966
			0.001	0.9972
			0.1	0.9972
		0.02	0	0.9965
			0.001	0.9971
			0.1	0.9971

Table 9. Fundamental frequency parameter $\bar{\omega}$ for free vibration of a thin-walled curved beam with one clamped and one free end for $N = 11$ and $\delta=1\times10^{-6}$; $\theta_0=45^0$.

θ_0	$\bar{C}$	ξ	$\bar{D}$	DQM
45°	0.05	0.005	0	0.7820
			0.001	0.9060
			0.1	0.9733
		0.02	0	0.7804
			0.001	0.9034
			0.1	0.9732
	2.0	0.005	0	0.9704
			0.001	0.9719
			0.1	0.9762
		0.02	0	0.9703
			0.001	0.9719
			0.1	0.9761

Table 10. Natural frequency ω (rad/s) for free vibration of a thin-walled curved beam with simply supported ends for $N = 11$ and $\delta=1\times10^{-6}$ neglecting rotatory inertia.

θ_0	$\omega(rad/s)$	
	Exact (Shore and Chaudhuri, 1972)	DQM
10°	16815	16795
20°	3928.1	3913.3
30°	1542.2	1536.2
40°	745.96	743.81
50°	406.78	406.01
60°	240.21	239.91
70°	150.12	149.90
80°	97.526	97.475
90°	65.219	65.195

Table 11. Natural frequency ω (rad/s) for free vibration of a thin-walled curved beam with simply supported ends for $N = 11$ and $\delta=1\times10^{-6}$ including warping deformation or neglecting warping deformation.

θ_0	$\omega(rad/s)$			
	Exact (Culver, 1967)	DQM	Exact (Culver, 1967)	DQM
10°	10615	10614	5340.2	5339.6
20°	3130.1	3129.9	2460.8	2460.0
30°	1361.1	1361.1	1241.4	1241.3
40°	690.63	690.61	655.10	655.09
50°	387.50	387.50	373.35	373.35
60°	232.87	232.87	226.35	226.35
70°	147.01	147.01	143.74	143.74
80°	96.214	96.215	94.473	94.473
90°	64.615	64.616	63.648	63.648

4~7, the quadrature solution results are found to correlate very well with the numerical results. It is seen that in general, the numerical results by the DQM are lower than those by the Rayleigh-Ritz method, and the difference of the numerical results between the two methods decreases as $\bar{C}$ increases. Tables 8 and 9 show the numerical results by the DQM for the case of clamped-free ends.

The following examples are considered for more detailed analysis. The second example has a constant radius of curvature of 326.136 cm and a variety of opening angles with $\theta_0=10^\circ\sim90^\circ$. Cross-sectional properties of the beam are:

A (cross-sectional area) = 92.9 cm^2, I_x = 11362 cm^4, I_y = 3817 cm^4, I_w = 555878 cm^6, and K_T = 1470.85 cm^4. Values used for the elastic modulus, shear modulus, and mass per unit length are: E = 200.1 GN/m^2, G = 77.3 GN/m^2, and m=7.31 $N\text{-}sec^2/cm^2$. The natural frequencies, ω (rad/s), determined by exact solutions (Shore and Chaudhuri, 1972; Culver, 1967) are compared with those by the DQM for the case of both ends simply supported. The results by Shore and Chaudhuri (1972) neglecting rotatory inertia and by Culver (1967) neglecting warping deformations are summarized in Tables 10 and 11, respectively. It is also seen that excellent agreement is achieved between the DQM solutions and the analytical solutions for the cases of including warping or neglection warping deformations.

Table 12. Natural frequency ω (*rad/s*) for free vibration of a thin-walled curved beam with simply supported ends for N = 11 and $\delta=1\times10^{-6}$ neglecting both warping deformation and rotatory inertia.

θ_0	Exact (Shore and Chaudhuri, 1972)	FEM (Chaudhuri and Shore, 1977)	DQM
75°	215.66	214.24	215.57

Table 13. Fundamental frequency parameter $\kappa=\omega^2 mR^4/GK_T$ for coupled twist-bending vibration of a curved beam with clamped ends for N = 11 and $\delta=1\times10^{-6}$ neglecting both warping deformation and rotatory inertia.

θ_0	$\bar{C}$	Exact (Ojalvo, 1962)	DQM
180°	0.005	47.60	47.60
	0.2	13.36	13.36
	0.5	6.334	6.334
	1.0	3.375	3.375
	1.625	2.134	2.131
270°	0.005	3.304	3.305
	0.2	1.646	1.646
	0.5	0.9548	0.9548
	1.0	0.5776	0.5779
	1.625	0.3939	0.3939
360°	0.005	0.4540	0.4541
	0.2	0.3350	0.3350
	0.5	0.2533	0.2533
	1.0	0.1915	0.1915
	1.625	0.1528	0.1528

Table 14. Fundamental frequency parameter $\kappa=\omega^2 mR^4/GK_T$ for coupled twist-bending vibration of a curved beam with simply supported ends for N = 11 and $\delta=1\times10^{-6}$ neglecting both warping deformation and rotatory inertia.

θ_0	$\bar{C}$	Exact (Rodgers and Warner, 1973)	DQM
90°	0.005	35.29	35.29
	0.2	20.00	20.01
	0.5	12.00	12.00
	1.0	7.200	7.200
	1.625	4.800	4.800

Chaudhuri and Shore (1977) also determined the natural frequencies of the following example using the FEM with 39 elements for the cases of simply supported ends, that neglected both warping deformations and rotatory inertia. The third example has a constant radius of curvature of 254.0 *cm* with θ_0 = 75°. Cross-sectional properties of the beam are: A = 322.6 cm^2, I_x = 17341.8 cm^4, I_y = 4335.43 cm^4, I_w = 0, and K_T = 11913.73 cm^4. Values used for the elastic modulus, shear modulus, and mass per unit length are: E = 207.0 GN/m^2, G = 79.6 GN/m^2, and m=25.4 $N\text{-}sec^2/cm^2$. The natural frequencies, ω (*rad/s*), determined by exact solutions (Shore and Chaudhuri, 1972) and by the FEM (Chaudhuri and Shore, 1977) are compared with those by the DQM for the case of both ends simply supported. The results are summarized in Table 12. Table 12 shows that the numerical results by the DQM using 11 discrete points are more accurate than those by the FEM using 39 elements.

Ojalvo (1962) and Rodgers and Warner (1973) studied the coupled twist-bending vibration of curved beams using closed-form solutions of the

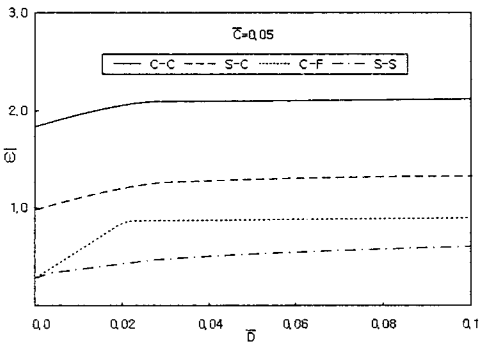

Fig. 2. Parametric results of free vibration of curved beams with N = 11 and $\delta=1\times10^{-6}$; θ_0 = 90°, ξ= 0.005, and $\bar{C}$ =0.05.

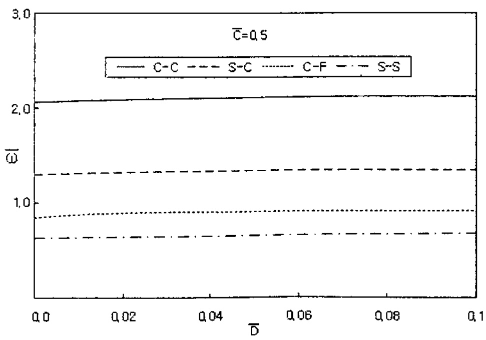

Fig. 3. Parametric results of free vibration of curved beams with N = 11 and $\delta=1\times10^{-6}$; θ_0 = 90°, ξ= 0.005, and $\bar{C}$ =0.5.

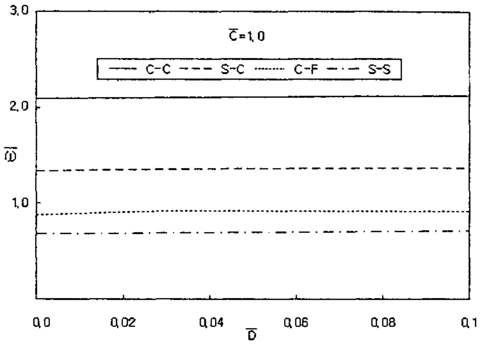

Fig. 4. Parametric results of free vibration of curved beams with $N = 11$ and $\delta=1\times10^{-6}$; $\theta_0 = 90°$, $\xi= 0.005$, and $\bar{C} = 1.0$.

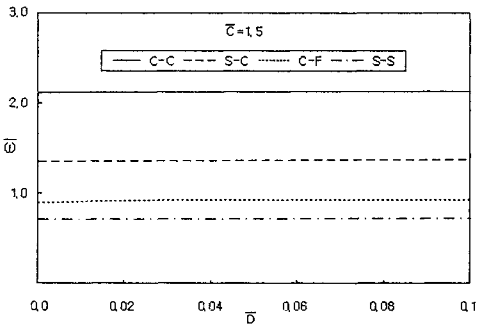

Fig. 5. Parametric results of free vibration of curved beams with $N = 11$ and $\delta=1\times10^{-6}$; $\theta_0 = 90°$, $\xi= 0.005$, and $\bar{C} = 1.5$.

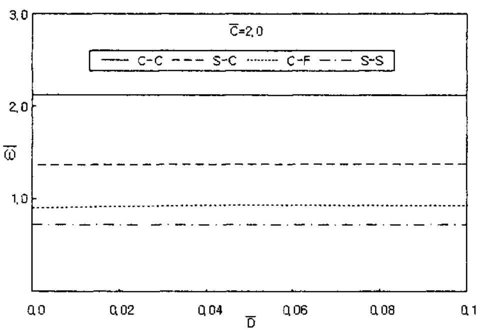

Fig. 6. Parametric results of free vibration of curved beams with $N = 11$ and $\delta=1\times10^{-6}$; $\theta_0 = 90°$, $\xi= 0.005$, and $\bar{C} = 2.0$.

equations of motion neglecting both rotatory inertia and warping deformations, given by Eqs. (1) and (2), and calculated the fundamental frequency parameters, $\kappa = \omega^2 mR^4 / GK_T$, for the cases of both ends clamped and simply supported, respectively. The results are summarized in Tables 13 and 14.

Finally, parametric results using the DQM are presented in dimensionless form in Figs. 2~6. The examples considered here have $\xi= 0.005$ and $\theta_0=90^0$ with clamped-clamped (C-C), simply supported-clamped (S-C), clamped-free (C-F), and simply supported-simply supported (S-S) ends.

The superior computational efficiency of the DQM over the numerical solution methods such as the finite difference, finite element, and other approximate methods is in indeed well established and has been dealt with in detail in other works; see, for example, Malik and Bert (1994) and Malik and Civan (1995).

6. Conclusions

The differential quadrature method (DQM) was used to compute the frequencies of free vibration of a thin-walled curved beam with various nondimensional parameters and boundary conditions. It can be seen that the DQM with only a rather small number of grid points yields numbers that compare very well with the exact results and the Rayleigh-Ritz results. The results by the DQM using 11 discrete points are more accurate than those by the FEM using 39 elements.

Finally, the results have been presented for a cantilever beam not previously considered for this problem, and the parametric results using the DQM are presented in dimensionless form. It is believed that the data would be useful to the researchers for the comparisons of their solutions in this area.

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Nomenclature

The following symbols are used in this paper:

A : Beam cross-sectional area;
A_{ij} : Weighting coefficients for the first derivatives;
B_{ij} : Weighting coefficients for the second derivatives;
B_w : Bimoment;
C_{ij} : Weighting coefficients for the third derivatives;
$\bar{C}, \bar{D}$: Parameters, Eq. (8);
D_{ij} : Weighting coefficients for the fourth derivatives;
E : Modulus of elasticity;
$f(x)$: General function;

$f(x_j)$	: Function value at point x_j;
G	: Shear modulus;
I_w	: Warping constant;
I_x, I_y	: Area moment of inertia about x-axis and y-axis, respectively;
K_T	: Saint-Venant torsion constant;
L	: Differential operator;
M_x	: Bending moment;
m	: Mass per unit length;
N	: Number of discrete points;
Q	: Transverse shear force;
R	: Horizontal radius of curvature;
r	: Polar radius of gyration;
T	: Torsional moment;
t	: Time;
u	: Radial displacement in x-direction;
$V(z), v(z, t)$	: Displacements in y-direction;
W_{ij}	: Weighting coefficients;
w	: Tangential displacement in z-direction;
X	: Dimensionless position coordinate;
x_j	: Discrete point in domain;
x, y, z	: Coordinate axes;
δ	: Small dimensionless distance measured from boundary ends of member;
θ	: Angle from left support to generic point;
θ_0	: Opening angle of member;
κ	: Fundamental frequency parameter $\omega^2 mR^4 / GK_T$;
ξ	: Parameter, Eq. (8);
τ	: Warping;
$\Phi(z), \phi(z, t)$	: Angles of twist;
ω	: Natural frequency (*rad/s*);
$\bar{\omega}$	: Fundamental frequency parameter, Eq. (8).

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